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HYPERBOLIC HEAT-CONDUCTION EQUATION. MIXED BOUNDARY-VALUE PROBLEM

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The integral representation of the exact solution of a mixed problem for a hyperbolic heat-conduction equation with the most general propositions within the framework of generalized thermomechanics has been obtained. The thermal state of a bounded medium with thermal effects at the boundary has been investigated numerically on a personal computer.

Keywords: *heat conduction, hyperbolic equation, thermomechanics, Laplace transformation, boundary-value problem.*

Recent years have seen a renewed interest in studying a hyperbolic heat-conduction equation [1, 2] first proposed in [3] and developed in [4]. This equation is used for description of temperature fields that occur in high-intensity heat exchange in impulse and laser equipment, in laser processing of materials, in plasma-spraying processes, in energy channels of nuclear reactors, and in disperse systems and granular materials.

Within the framework of a generalized heat-conduction theory, the problem on the structure of temperature fields leads us to seek, in the domain $D = \{(\tau, z)\colon 0 \le \tau \le \tau_1 < \infty,\ 0 \le z \le l\}$, a fairly smooth bounded solution of the equation

$$b_0^2 \frac{\partial^2 u}{\partial \tau^2} + b_1^2 \frac{\partial u}{\partial \tau} + b_2^2 u - \frac{\partial^2 u}{\partial z^2} = f(\tau, z) \quad (1)$$

from the initial conditions

$$u|_{\tau=0} = g_1(z), \quad \left.\frac{\partial u}{\partial \tau}\right|_{\tau=0} = g_2(z) \quad (2)$$

and the boundary conditions

$$\left.\left[(-1)^j h_{j1} \frac{\partial u}{\partial z} + h_{j2} \frac{\partial u}{\partial \tau} + h_{j3} u\right]\right|_{z=R_j} = \omega_j(\tau), \quad R_1 = 0, \quad R_2 = l, \quad j = 1, 2. \quad (3)$$

We assume that: 1) the constants are $h_{jk} \ge 0$ and $h_{j1} + h_{j2} + h_{j3} \ne 0$; 2) the boundary of the domain is soft ($h_{12} > 0$ and $h_{22} > 0$ or at least one of these numbers is positive); 3) the prescribed and sought functions are the inverse Laplace transforms of the time variable τ. Then the solution of problem (1)–(3), which has been constructed by the method of integral Laplace transformation, can be represented in the form of parabolic, zero, and hyperbolic parts [5]:

$$u(\tau, z) = \int_0^\tau W_1(\tau - s, z)\,[\omega_1(s) + h_{12} g_1(0)\,\delta_+(s)]\,ds$$

$$+ \int_0^\tau W_2(\tau - s, z)\,[\omega_2(s) + h_{22} g_1(l)\,\delta_+(s)]\,ds + \int_0^\tau \int_0^l E(\tau - s, z, \xi) f_1(s, \xi)\,d\xi ds$$

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$$+ b_0^2 \frac{\partial}{\partial \tau} \int_0^l E(\tau, z, \xi)\, g_1(\xi)\, d\xi + \int_0^l E(\tau, z, \xi) \left[b_0^2 g_2(\xi) + b_1^2 g_1(\xi) \right] d\xi .$$

Here, we have used the following functions:

$$W_i(\tau, z) = W_i^{\mathrm{p}}(\tau, z) + W_i^{\mathrm{z}}(\tau, z) + W_i^{\mathrm{h}}(\tau, z) ,$$

$$W_i^{\mathrm{p}}(\tau, z) = 2 \sum_{n=1}^{n_0-1} \left[A_{ij}(\beta_n, z) \cos \bar{\bar{b}}_n \tau - B_{ij}(\beta_n, z)\, \bar{\bar{b}}_n \sin \bar{\bar{b}}_n \tau \right] \exp(-k\tau)$$

$$\text{when} \quad \beta_n > \sqrt{\frac{b_1^4}{4b_0^4} - b_2^2} \quad (b_n^2 < 0) \quad \text{for} \quad i = 1, j = 2 ; \quad \text{for} \quad i = 2, j = 1 ;$$

$$A_{12}(\beta_n, z) = d^{-1} \left[\varphi_{21}(\beta_n, z)\, \chi_{1n} - b_n^2 \varphi_{22}(\beta_n, z)\, \chi_{2n} \right]; \quad B_{12}(\beta_n, z) = d^{-1} \left[\varphi_{22}(\beta_n, z)\, \chi_{1n} - \varphi_{21}(\beta_n, z)\, \chi_{2n} \right];$$

$$A_{21}(\beta_n, z) = d^{-1} \left[\varphi_{11}(\beta_n, z)\, \chi_{1n} - b_n^2 \varphi_{12}(\beta_n, z)\, \chi_{2n} \right]; \quad B_{21}(\beta_n, z) = d^{-1} \left[\varphi_{12}(\beta_n, z)\, \chi_{1n} - \varphi_{11}(\beta_n, z)\, \chi_{2n} \right];$$

$$d = \chi_{1n}^2 - b_n^2 \chi_{2n}^2 , \quad \varphi_{11}(\beta_n, z) = h_{11} \beta_n \cos \beta_n z + (h_{13} - k h_{12}) \sin \beta_n z ; \quad \varphi_{12}(\beta_n, z) = h_{12} \bar{b}_n \sin \beta_n z ;$$

$$\varphi_{22}(\beta_n, z) = h_{22} \bar{b}_n \sin(\beta_n (l - z) ; \quad \varphi_{21}(\beta_n, z) = h_{21} \beta_n \cos(l - z)\, \beta_n + (h_{23} - k h_{22}) \sin \beta_n (l - z) ;$$

$$\chi_{1n} = \psi_2^*(\beta_n) - k_4 b_1^2 b_0^{-2} \sin \beta_n l - \psi_3^*(\beta_n)\, b_n^2 (4\beta_n b_0^2)^{-1} + b_1^2 b_n^2 k_4 l\, (4\beta_n^2 b_0^4)^{-1} \cos \beta_n l ;$$

$$\chi_{2n} = (2\beta_n)^{-1} \left(\psi_1^*(\beta_n) + \psi_2^*(\beta_n) + \psi_3^*(\beta_n)\, b_1^2 (4\beta_n b_0^2)^{-1} \right) + b_0^{-2} k_4 \sin \beta_n l - k_4 l\, (b_1^4 + \beta_n^2)(8\beta_n b_0^4)^{-1} \cos \beta_n l ;$$

$$\psi_1^*(\beta_n) = 2k_1 \beta \sin \beta l - \left[l(-k_1 \beta^2 + k_2) + k_3 \right] \cos \beta l + k_3 l \beta \sin \beta l ;$$

$$\psi_2^*(\beta_n) = k_5 \sin \beta l + k_6 \beta \cos \beta l ; \quad \psi_3^*(\beta_n) = (l k_5 + k_6) \cos \beta l - k_6 l \beta \sin \beta l ;$$

$$W_i^{\mathrm{z}}(\tau, z) = 2 \frac{\varphi_{j1}(\beta_n, z)}{\chi_{1n}} \exp(-k\tau) \quad \text{when} \quad \beta_n = \sqrt{\frac{b_1^4}{4b_0^4} - b_2^2} \quad (b_n^2 = 0) ;$$

$$W_i^{\mathrm{h}}(\tau, z) = 2 \sum_{n=n_0+1}^{\infty} \left[A_{ij}(\beta_n, z) \cosh \bar{b}_n \tau + B_{ij}(\beta_n, z)\, b_n \sinh \bar{b}_n \tau \right] \exp(-k\tau) \quad \text{when} \quad \beta_n < \sqrt{\frac{b_1^4}{4b_0^4} - b_2^2} \quad (b_n^2 > 0) ;$$

$$E(\tau, z, \xi) = E^{\mathrm{p}}(\tau, z, \xi) + E^{\mathrm{z}}(\tau, z, \xi) + E^{\mathrm{h}}(\tau, z, \xi) ,$$

$$E^{\mathrm{p}}(\tau, z, \xi) = 2 \sum_{n=1}^{n_0-1} \left[C_{12}(z, \xi) \cos \bar{\bar{b}}_n \tau - C_{21}(z, \xi)\, \bar{\bar{b}}_n \sin \bar{\bar{b}}_n \tau \right] \exp(-k\tau) \quad \text{when} \quad b_n^2 < 0 ;$$

$$C_{12}(z, \xi) = \left(Y_{1n}(z, \xi)\, \chi_{1n} - b_n^2 Y_{2n}(z, \xi)\, \chi_{2n} \right) (d\beta_n)^{-1} ;$$

$$C_{21}(z,\xi) = \left(Y_{2n}(z,\xi)\,\chi_{1n} - Y_{1n}(z,\xi)\,\chi_{2n}\right)(d\beta_n)^{-1} ;$$

$$E^z(\tau, z, \xi) = 2\,\frac{Y_{1n}(z,\xi)}{\chi_{1n}\beta_n}\exp(-k\tau) \quad \text{when} \quad b_n^2 = 0 ;$$

$$E^h(\tau, z, \xi) = 2\sum_{n=n_0+1}^{\infty}\left[C_{12}(z,\xi)\cosh \bar{b}_n\tau - C_{21}(z,\xi)\,b_n \sinh \bar{b}_n\tau\right]\exp(-k\tau) \quad \text{when} \quad b_n^2 > 0 ;$$

$$Y_{1n}(z,\xi) = \varphi_{11}(\beta_n, z)\,\varphi_{21}(\beta_n, z) + b_n^2\varphi_{12}(\beta_n, z)\,\varphi_{22}(\beta_n, z) ;$$

$$Y_{2n}(z,\xi) = \varphi_{11}(\beta_n, z)\,\varphi_{22}(\beta_n, z) + \varphi_{12}(\beta_n, z)\,\varphi_{21}(\beta_n, z) ;$$

β_n are the roots of the generalized transcendental equation

$$z_1^+(\beta)\sin\beta l + z_2^+(\beta)\,\beta\cos\beta l = 0 .$$

In the above formulas, we have adopted the following notation:

$$k = b_1^2\left(2b_0^2\right)^{-1}, \quad \bar{b}(\beta) = \left(2b_0^2\right)^{-1}\left(b_1^4 - 4b_0^2\left(b_2^2 + \beta^2\right)\right)^{1/2} = \left(2b_0^2\right)^{-1} b(\beta) ,$$

$$b_n = \sqrt{b_1^4 - 4b_0^2\,(b_2^2 + \beta_n^2)} , \quad \bar{b}_n = \left(2b_0^2\right)^{-1} b_n , \quad b_n = \sqrt{\beta_n^2 - \left[\frac{b_1^4}{4b_0^4} - b_2^2\right]} ,$$

$$z_1^+(\beta) = h_1(P_+)\,h_2(P_+) - h_{11}h_{21}\beta^2 , \quad z_2^+(\beta) = h_{11}h_2k_2(P_+) + h_{21}h_1(P_+) , \quad P_+(\beta) = -k + \bar{b}(\beta) ,$$

$$k_1 = h_{11}h_{21} , \quad k_2 = h_{13}h_{23} , \quad k_3 = h_{11}h_{23} + h_{21}h_{13} , \quad k_4 = h_{12}h_{22} , \quad k_5 = h_{12}h_{23} + h_{13}h_{22} ,$$

$$k_6 = h_{11}h_{22} + h_{12}h_{21} , \quad h_j(P_+) = h_{j2}P_+ + h_{j3} ,$$

$\delta_+(s)$ is the delta function concentrated at the point $s = 0_+$ and $j = 1$ and 2.

We give the cases of the most frequent practical occurrence. Let the lines $z = 0$ and $z = l$ be rigid in relation to the reflection of waves ($h_{12} = 0$ and $h_{22} = 0$). Then we directly find: $k_4 = 0$, $k_5 = 0$, $k_6 = 0$, $\Psi_2^* = 0$, $\Psi_3^* = 0$, $\chi_{1n} = 0$, $\chi_{2n} = (2\beta_n)^{-1}\Psi_1^*(\beta_n) \equiv (2\beta_n)^{-1}\{(2k_1 + k_3 l)\beta_n \sin\beta_n l - [l(k_2 - k_1\beta_n^2) + k_3]\cos\beta_n l\}$, $\varphi_{12}(\beta_n, z) = 0$, and $\varphi_{22}(\beta_n, z) = 0$. The principal solutions (Green's function and the fundamental function) have the form

$$W_i(\tau, z) = 2\sum_{n=1}^{\infty}\frac{\bar{\varphi}_{j1}(\beta_n, z)}{(2\beta_n)^{-1}\Psi_1^*(\beta_n)}\,\frac{\sinh \bar{b}_n\tau}{b_n}\exp(-k\tau) \quad \text{at} \quad i = 1,\ j = 2 ; \quad \text{at} \quad i = 2,\ j = 1 ;$$

$$E(\tau, z, \xi) = 4\sum_{n=1}^{\infty}\frac{\bar{\varphi}_{11}(\beta_n, z)\,\bar{\varphi}_{21}(\beta_n, \xi)}{\Psi_1^*(\beta_n)}\,\frac{\sinh \bar{b}_n\tau}{b_n}\exp(-k\tau) .$$

In these equalities, $\bar{\varphi}_{11} = h_{11}\beta_n \cos\beta_n z + h_{13}\sin\beta_n z$ and $\bar{\varphi}_{21} = h_{21}\beta_n \cos(l - z)\beta_n + h_{23}\sin\beta_n(l - z)$, and β_n are the roots of the transcendental equation

$$\left(h_{13}h_{23} - h_{11}h_{21}\beta^2\right)\sin\beta l + (h_{11}h_{23} + h_{21}h_{13})\,\beta\cos\beta l = 0 .$$

If the boundary conditions of the 1st kind ($h_{11} = 0$, $h_{21} = 0$, and $h_{13} = h_{23} = 1$) are specified at the boundary of the domain, we have $k_1 = 0$, $k_2 = 1$, $k_3 = 0$, $\psi_1^*(\beta_n) = -l \cos \beta_n l$, $\overline{\varphi}_{11}(\beta_n, z) = \sin \beta_n z$, and $\overline{\varphi}_{21}(\beta_n, z) = \sin \beta_n (l - z)$, and the transcendental equation $\sin \beta l = 0$ has roots $\beta_n = n\pi/l$. Then we obtain

$$\psi_1^* (\beta_n) = - l (- 1)^n , \quad \varphi_{11} = \sin \beta_n z , \quad \varphi_{21} = - (- 1)^n \sin \beta_n z ,$$

$$W_i (\tau, z) = \frac{4}{l} \sum_{n=1}^{\infty} (- 1)^{(n+1)j} \beta_n \sin \beta_n z \frac{\sinh \overline{b}_n \tau}{b_n} \exp (- k\tau) \quad \text{when} \quad i = 1, \; j = 0 ; \quad \text{when} \quad i = 2 , \; j = 1 ;$$

$$E (\tau, z, \xi) = \frac{4}{l} \sum_{n=1}^{\infty} \sin \beta_n \xi \sin \beta_n z \frac{\sinh \overline{b}_n \tau}{b_n} \exp (- k\tau) .$$

If the boundary conditions of the 2nd kind ($h_{11} = h_{21} = 1$ and $h_{13} = h_{23} = 0$) are specified at the boundary $z = 0$ and $z = l$, the transcendental equation $\beta^2 \sin \beta l = 0$ has roots $\beta_{n0} = 0$ and $\beta_n = n\pi/l$. In this case we have $W_i(\tau, z) = \frac{4}{l} \sum_{n=0}^{\infty} \varepsilon_n \; (-1)^{nj} \cos \beta_n z \frac{\sinh \overline{b}_n \tau}{b_n} \exp (-k\tau)$ for $i = 1$ and $j = 0$; for $i = 2$ and $j = 1$, and for $\varepsilon_0 = 1/2$, $\varepsilon_1 = \varepsilon_2 = ... = \varepsilon_n = ... = 1$ and $\beta_0 = 0$;

$$E (\tau, z, \xi) = \frac{4}{l} \sum_{n=1}^{\infty} \varepsilon_n \cos \beta_n \xi \cos \beta_n z \frac{\sinh \overline{b}_n \tau}{b_n} \exp (- k\tau) .$$

We investigate the thermal state with allowance for the finite velocity of propagation of heat of the following model problem:

$$\frac{1}{c_q^2} \frac{\partial^2 u}{\partial \tau^2} + \frac{1}{a} \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial z^2} , \quad u (z, 0) = c_0 , \quad \frac{\partial u (z, 0)}{\partial \tau} = 0 , \quad u (0, \tau) = c_3 , \quad u (l, \tau) = c_2 ; \quad c_0, c_3, c_2 = \text{const} .$$

It is noteworthy that in this problem we are able to additionally separate the elliptic part $\left(c_3 - \frac{c_3 - c_2}{l} z \right)$ of the solution, which will be represented in the form

$$u (z, \tau) = c_3 - \frac{c_3 - c_2}{l} z + 2 \sum_{n=1}^{\infty} \left[\frac{c_0 - c_3 - (c_0 - c_2) (- 1)^n}{n\pi} \sin \frac{n\pi z}{l} \exp \left(- \frac{c_q^2}{2a} \right) \right.$$

$$\left. \times \left(\frac{\sinh \frac{c_q^2}{2a} \sqrt{1 - \frac{4n^2 \pi^2 a^2}{l^2 c_q^2}} \tau}{\sqrt{1 - \frac{4n^2 \pi^2 a^2}{l^2 c_q^2}}} + \cosh \frac{c_q^2}{2a} \sqrt{1 - \frac{4n^2 \pi^2 a^2}{l^2 c_q^2}} \tau \right) \right] . \quad (4)$$

TABLE 1. Temperature of the Bounded Medium with a Varying Number of Terms in the Series (ξ = 5)

M^2	t	n									
		1	2	3	4	5	10	50	51	100	1000
1.846	41	0.49950	0.49950	0.49964	0.49964	0.49958	0.49961	0.49959	0.49960	0.49960	0.49966
1.29	15	0.45925	0.45925	0.46967	0.46967	0.46668	0.46830	0.46691	0.46694	0.46689	0.46702
1	12	0.46990	0.46990	0.47714	0.47714	0.47556	0.47647	0.47568	0.47532	0.47558	0.47549
0	6	0.14787	0.14787	0.14890	0.14890	0.14890	0.14890	0.14890	0.14890	0.14890	0.14890

TABLE 2. Time of Transition of the Temperature to a Stationary State

M^2	ξ									
	1	4	5	6	7	8	8.5	9	9.5	9.8
0	61	72	73	72	71	68	65	61	54	45
0.5	11	11	11	10	10	10	9	9	8	7
1	22	22	21	21	20	19	19	18	16	14
1.29	28	28	28	27	26	25	25	24	22	19
1.846	42	41	41	40	38	37	36	35	35	29
2.3	53	52	52	51	48	46	46	44	41	37

TABLE 3. Temperature Distribution in a Fixed Time Section

M^2	ξ											
	0.5	0.8	1	1.5	2	3	4	5	6	7	8	9
0	0.942	0.907	0.883	0.825	0.768	0.656	0.549	0.446	0.349	0.256	0.168	0.083
0.5	0.950	0.920	0.900	0.850	0.800	0.700	0.600	0.500	0.400	0.300	0.200	0.100
1	0.950	0.920	0.900	0.850	0.800	0.700	0.600	0.500	0.400	0.300	0.200	0.100
1.29	0.949	0.919	0.899	0.848	0.798	0.698	0.599	0.499	0.399	0.299	0.200	0.100
1.846	0.937	0.899	0.874	0.817	0.768	0.675	0.577	0.482	0.385	0.289	0.192	0.096
2.3	0.893	0.834	0.798	0.729	0.686	0.608	0.517	0.434	0.345	0.260	0.172	0.087

TABLE 4. Temperature Distribution in a Fixed Coordinate Section

M^2	t											
	10	11	13	15	18	23	26	30	35	40	42	45
0	0.263	0.285	0.324	0.355	0.392	0.434	0.451	0.467	0.480	0.488	0.490	0.493
0.5	0.445	0.463	0.484	0.493	0.498	0.500	0.500	0.500	0.500	0.500	0.500	0.500
1	0.445	0.463	0.484	0.493	0.495	0.500	0.500	0.500	0.500	0.500	0.500	0.500
1.29	0.346	0.385	0.437	0.467	0.487	0.498	0.499	0.500	0.500	0.500	0.500	0.500
1.846	0.050	0.123	0.241	0.326	0.407	0.469	0.486	0.494	0.498	0.499	0.500	0.500
2.3	0.000	0.000	0.020	0.128	0.268	0.400	0.441	0.471	0.489	0.496	0.497	0.498

TABLE 5. Temperature Distribution in the Vicinity of the Bound of Variation in the Temperature

M^2	ξ									
	0.01	0.02	0.05	0.08	0.1	0.12	0.14	0.15	0.17	0.2
0	0.99883	0.99766	0.99415	0.99064	0.98830	0.98596	0.98362	0.98246	0.98012	0.97661
0.5	0.99900	0.99800	0.99500	0.99200	0.99000	0.98900	0.98600	0.98500	0.98300	0.98000
1	0.99900	0.99800	0.99499	0.99199	0.98999	0.98799	0.98599	0.98499	0.98258	0.97998
1.29	0.99899	0.99798	0.99495	0.99194	0.98994	0.98793	0.98593	0.98493	0.98291	0.97998
1.846	0.99915	0.99827	0.99531	0.99172	0.98910	0.98650	0.98408	0.98298	0.98101	0.97844
2.3	0.99777	0.99563	0.99051	0.98792	0.98723	0.98661	0.98534	0.98430	0.98128	0.97485

When $c_q \to \infty$ we obtain the result existing in the literature [6], which confirms the correctness of the model in question. We pass to dimensionless coordinates:

$$T = \frac{u - c_0}{c_3 - c_0}\,, \quad t = \frac{c^2 \tau c_T \rho}{\lambda}\,, \quad M = \frac{c}{c_q}\,, \quad \xi = \frac{czc_T\rho}{\lambda}\,, \quad l_0 = \frac{clc_T\rho}{\lambda}\,.$$

Certain practically important experiments have been calculated on a personal computer in the Mathcad 200 Professional system from the formula of the solution (4) in dimensionless coordinates ($c_0 = c_2 = 0$, $c_3 = 1$, and $l_0 = 10$). Table 1 gives the values of temperature accurate to five significant digits; from this table, it is clear that up to ten terms of the series are sufficient for calculation.

Table 2 gives the time of transition of the temperature to a stationary state accurate to three significant digits ($n = 10$); the transition time increases with M. With distance from the boundary exposed to temperature, the stationary-state transition is the more rapid, and the larger is the distance to the boundary, which is attributed to the presence of the finite velocity of propagation of heat. In the parabolic case the transition time comes later by virtue of the infinite velocity of propagation of heat.

The temperature values given in Table 3 (three significant figures, $n = 10$, $t = 25$) for different ξ and M and in Table 4 (three significant digits, $\xi = 5$, $n = 5$) for different t and M show that the temperature drops in fixed cross sections in the medium with increase in M, since the velocity of propagation of heat decreases.

As follows from Table 5, we observe the difference in the temperatures obtained in solving the parabolic and hyperbolic equations in the third decimal digit with change in M and ξ ($n = 5$, $t = 25$, five significant digits).

The investigations carried out show that the solutions of the hyperbolic and parabolic heat-conduction equations are virtually coincident for large times; the differences are found only at the initial instants of time.

NOTATION

a, thermal diffusivity; $a^* = a(1 + a\chi^2\tau_r)^{-1}$; $b_0^2 = 1/c_q^2$, $b_1^2 = 1/a^*$, and $b_2^2 = \chi^2$; $c_q = \sqrt{a/\tau_r}$, velocity of propagation of heat; c, velocity of propagation of the longitudinal wave; c_T, specific heat at constant temperature T; h_{j1}, h_{j2}, and h_{j3}, coefficients of connection of the boundary conditions of the 1st, 2nd, and 3rd kind; u, temperature; z, axis of the Cartesian coordinate system; α_z, coefficient of heat transfer from the surface $z = \pm\delta$; α_R, coefficient of heat transfer from the surface $r = R$; λ, thermal conductivity; ρ, density; τ, time variable; τ_r, relaxation time; $\chi^2 = \alpha_z(\lambda\delta)^{-1}$, for a plate; $\chi^2 = \alpha_R(\lambda R)^{-1}$, for a cylindrical rod [4]. Subscripts and superscripts: h, hyperbolic; z, zero; p, parabolic; q, heat flux; r, relaxation.

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